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Full Research Paper

Computational Extraction of Legal Causes via al-Ṣabr wa al-Taqsim

A Set-Theoretic Formalization for Closed Fiqh Chapters

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ABSTRACT

This paper presents a set-theoretic formalization of the classical usuli method of al-Ṣabr wa al-Taqsim (Examination and Division) for extracting legal causes (‘*‘ilal*) within closed chapters of jurisprudence. A computational algorithm is introduced that extracts minimal operational rules from a truth table of juristic verdicts. The principal result is that, given a complete truth table for a closed chapter, the algorithm computes the minimal structural generators of the ruling and eliminates all logically redundant attributes. The resulting structures constitute admissible candidate causes for subsequent juristic evaluation.

Keywords: Islamic jurisprudence, Usul al-Fiqh, al-Ṣabr wa al-Taqsim, legal causation, candidate *‘illah*, closed chapters, truth tables, Boolean minimization, complete induction, computational fiqh

1. Introduction

Traditional inference in Islamic jurisprudence (fiqh) often proceeds through case-based reasoning, textual interpretation, analogical extension, and forms of induction that are not usually exhaustive. In addition to this, and due in part to the influence of important fiqh scholars such as al-Ghazali and Averroes, juristic rulings were also reached using classical deductive Aristotelian and Avicennian logic [4]. Toward the end of his life, al-Ghazali explicitly stressed the importance of deductive logic-based reasoning, treating logic as a general introduction to the sciences and as a condition for trustworthy scientific inquiry [5].

The present paper studies a restricted setting in which both analogical and deductive reasoning mechanisms can be represented by a finite conceptual model. For the purposes

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of the formal model, we consider a school-relative corpus of legal source material and fiqh-ontological definitions as fixed. Relative to such a fixed corpus, the paper asks whether a given chapter can be represented by a finite set of legally operative concepts and a complete ruling function over those concepts.

If the relevant source corpus, ontology vocabulary, and operative legal rules are treated as fixed relative to a given school, a further question arises: where does the apparent open-endedness of fiqh application come from? How can a finite ontological structure govern indefinitely many concrete cases?

The answer is twofold. First, the open-ended application of a legal rule emerges from the contexts in which that rule is applied to concrete world situations. Second, the premises of a legal rule, although fixed within the model, may apply to indefinitely many situations because the categories used in the sources are abstract. For example, in the chapter of Tahara (purification), many real-world situations may fall under the abstract concept of an "excuse" (*'Udhr*), which serves as a premise for the rule of Tayammum. While keeping the rule fixed and binding its premises to this abstract notion, one can map indefinitely many concrete situations to it. We call the structured set of such abstract legally operative notions for a given fiqh chapter the **Ontological Concepts Tree (OCT)**.

This leads to the epistemological distinction between Avicennian/Aristotelian Logic and Fiqhi Qiyas, which governs how the OCT operates:

- **Avicennian/Aristotelian Logic** relies on deductive syllogisms where the middle term is an ontological cause. The connection between premise and conclusion is one of strict necessity; if the cause is present, the effect must physically follow. The rule $A \Rightarrow B$ admits no exceptions (monotonic).
- **Fiqhi Qiyas** relies on analogical reasoning where the middle term is a teleological/legal cause (*'Illah*). The connection yields probabilistic knowledge and operates as a default rule; the Lawgiver may suspend the ruling due to a competing legal objective or preventative (*Mani'*). Thus, $A \Rightarrow B$ is non-monotonic: the presence of a preventative M can yield $A \wedge M \Rightarrow \neg B$.

To formalize this system computationally and logically, we must introduce strict methodological assumptions. The most critical is the **School-Relative Approach**. We do not treat the OCT as a subject of philosophical dispute or semantic fuzziness. Instead, we adopt the **Closed-World Assumption (CWA)**: the OCT is strictly bounded by the syntactic and semantic definitions explicitly laid out in the primary reference text (*Matn* or *Mukhtasar*) of a specific school of thought (*Madhhab*). By anchoring the OCT to a specific school's text, the relevant meanings are treated as fixed relative to the selected school text.

A Fiqh chapter for which the school-relative OCT is constant is defined as a **Closed Chapter**. Under this assumption, the combinatorics behind the chapter, even when exhaustive, yield a finite, constant state space. Because the number of OCT nodes (n) and the valid preventatives (M) are strictly finite and bounded by the text, the total number of possible states is 2^n.

This has two methodological consequences.

1. **Shift to Complete Induction**: For a closed chapter, we can transition the mode of Fiqh inference from incomplete induction to complete induction. The finite state space allows for the exhaustive construction of a truth table over the OCT. The non-monotonicity of Fiqhi Qiyas can be represented as bounded defeasibility within this table (where rows containing preventatives override default rows).

2. **Structural Comparison of Madhhabs**: By formalizing the OCTs and their respective inference rules on a chapter-by-chapter basis, we gain a rigorous, practical method for comparing different schools of thought. Instead of comparing only isolated rulings or relying solely on informal semantic comparison, we can computationally compare the structural logic of different Madhhabs: we can map the differences in their OCTs, analyze how different schools define preventatives (M), and identify where their truth tables diverge.

In summary, the Closed Chapter Assumption does not claim to capture the infinite metaphysical essence of the law; rather, it rigorously models the internal logic of a Madhhab as it stands. This makes systematic structural comparison across legal schools more explicit.

Some of the practical benefits of this approach are that it simplifies the extraction of legal causes (‘*‘ilal*), a task among the central tasks of Usul al-Fiqh. This paper does not attempt to prove ontological fiqh-causation. Rather, it formalizes the structural extraction of candidate causes from a complete operational model represented as a truth table.

The present paper should also be read as a domain-specific continuation of earlier work on logically complete ontology-level computation, where ontological term systems, decision components, inductive components, deductive components, and rational response components were proposed for relational database systems [1]. Here, the same general idea is specialized to closed fiqh chapters: the ontology is not a database ontology, but a school-relative Ontological Concepts Tree (OCT), and the response component is not a database query response, but the extraction of structurally admissible candidate *‘ilal*.

The framework presented here is a modelling framework: it applies when the relevant chapter can be projected, as an approximation, into a finite, school-relative concept vocabulary and a complete ruling table over that vocabulary.

2. Foundational Assumptions

2.1 Conceptual Representation of Juristic Reality

The present framework is based upon the observation that fiqh does not operate directly upon physical objects, but rather upon legally operative concepts abstracted from those objects. A juristic ruling is therefore determined not by the physical identity of a case, but by the conceptual attributes recognized as legally relevant within the chapter.

Axiom 1 (Conceptual Representation Principle)**.** *Let Ω denote the set of all real-world cases and let $V = \{v_1, v_2, \ldots, v_n\}$ denote the set of legally operative concepts of a fiqh chapter. There exists a mapping $\phi : \Omega \to \{0,1\}^n$ such that every ruling depends exclusively upon $\phi(\omega)$ for each $\omega \in \Omega$, and not upon any additional physical characteristics of the case.*

The mapping ϕ may be viewed as an Ontological Concepts Tree (OCT) encoding that transforms infinitely many real-world situations into a finite conceptual representation.

2.2 Conceptual Finiteness

The existence of a finite concept vocabulary immediately induces a finite legal state space.

Theorem 2 (Conceptual Finiteness)**.** *If $V = \{v_1, v_2, \ldots, v_n\}$ is finite, then the legal state space is finite.*

Proof. By Axiom 1, every legal case is represented by a valuation of the variables in V. Since each variable is binary, the number of distinct represented legal states is at most

$|\{0,1\}^n| = 2^n$. Because $n < \infty$, the quantity 2^n is finite. Therefore the legal state space is finite. □

The significance of this theorem is that it separates the apparent infinity of real-world instances from the finite conceptual structure upon which juristic reasoning actually operates.

2.3 Physical and Conceptual Closure

The present framework does not require the physical world itself to be finite. Instead it requires only the stability of the legally operative conceptual vocabulary.

Definition 3 (Physical Closure). *A domain is physically closed iff no new real-world entities can ever arise.*

Definition 4 (Conceptual Closure). *A domain is conceptually closed iff no future legally operative concept exists outside the concept set V.*

Remark 5. *Physical closure is generally impossible. Conceptual closure may nevertheless hold because infinitely many future cases can often be represented using a fixed conceptual vocabulary.*

2.4 Closed Chapters

The notion of a Closed Chapter is the foundational assumption underlying the remainder of the paper.

Definition 6 (Closed Chapter). *A fiqh chapter is said to be closed iff:*

1. *its legally operative concepts are represented by a finite set $V = \{v_1, v_2, \ldots, v_n\}$;*
2. *every future case admits a representation $\phi(\omega) \in \{0,1\}^n$;*
3. *no future legally operative concept exists outside the set V.*

A Closed Chapter therefore represents a conceptually complete operational model of the chapter under investigation.

2.5 Operational Closure and Conceptual Stability

The purpose of closure is not to make a metaphysical claim about reality. Rather, closure serves as an operational criterion governing whether the conceptual vocabulary of the chapter remains sufficient.

Principle 7 (Conceptual Stability). *A chapter may be treated as operationally closed whenever all newly encountered cases can be represented using the existing concept set V.*

Under this principle, the appearance of a new physical object or circumstance does not invalidate closure provided that the object can be expressed through already existing legally operative concepts.

2.6 Preservation and Failure of Closure

The following theorem characterizes when closure remains valid.

Theorem 8 (Closure Preservation). *Let C be a Closed Chapter with concept set V. Suppose a new case ω is encountered. If $\phi(\omega) \in \{0,1\}^n$, then the chapter remains closed.*

Proof. Since $\phi(\omega)$ is expressible using the existing concept set V, no new legally operative concept has been introduced. Therefore the conceptual vocabulary remains unchanged. Hence closure is preserved. □

The converse theorem identifies the precise failure condition of the framework.

Theorem 9 (Closure Failure)**.** *Let C be a Closed Chapter with concept set V. Suppose a newly encountered case requires a concept $x \notin V$. Then the Closed Chapter Assumption is false.*

Proof. The existence of $x \notin V$ contradicts Condition (3) of the Closed Chapter definition. Therefore the chapter is not closed. □

This theorem provides a falsification criterion for the framework: whenever a genuinely new legally operative concept is discovered, the chapter must be re-opened and the model reconstructed.

2.7 Complete Induction over Closed Chapters

The principal consequence of closure is that it transforms juristic analysis from incomplete induction into complete induction.

Corollary 10 (Complete Induction over a Closed Chapter)**.** *If a chapter is closed, then exhaustive enumeration of all legal states is possible.*

Proof. By Theorem 2, the represented legal state space contains at most 2^n distinct states. Since this quantity is finite, every state can be enumerated. Therefore complete induction over the chapter is possible. □

Consequently, the logical foundation of the present framework is the following chain:

$$\begin{aligned} &\text{Conceptual Representation} \Rightarrow \text{Conceptual Finiteness} \\ \Rightarrow\ &\text{Closed Chapter} \Rightarrow \text{Finite State Space} \\ \Rightarrow\ &\text{Complete Induction.} \end{aligned}$$

All subsequent algorithms and correctness proofs depend upon this foundational sequence.

2.8 Modular Decomposition and Hierarchical Truth Tables

The Closed Chapter Assumption guarantees that each chapter's legally operative concept set V is finite and fixed. However, a complex chapter such as *Ḥajj* (pilgrimage) may involve dozens of potentially relevant concepts if flattened into a single truth table. This would lead to a large n and a combinatorial state space of 2^n rows—still finite, but impractical to construct manually.

This hierarchical separation between lower-level factual representation and higher-level ontological/legal response is analogous to the earlier distinction between database-level and ontology-level processing in logically complete relational systems [1].

Fortunately, fiqh chapters are **modular by design**. A higher-level chapter does not operate directly on the atomic concepts of its sub-chapters; instead, it relies on the *outcome rulings* of those sub-chapters as atomic inputs. For example:

- The **Ḥajj** chapter uses the ruling of **Ṭahāra** (purification) as a single binary condition: "Is the pilgrim in a valid state of purity (*ṭāhir*)?"

- The **Ṣalāh** chapter similarly uses "ṭāhir" as a condition, but Ṣalāh itself may be a sub-chapter of Ḥajj (e.g., the prayer of *ṭawāf*).

This modular structure allows us to apply a **divide-and-conquer** strategy:

1. For each atomic closed sub-chapter (e.g., Ṭahāra), construct its complete truth table over its own small variable set V_{sub}.
2. Compute the ruling function h_{sub} from that table using the algorithm of Section 3.
3. Treat h_{sub} as a **single binary variable** in the truth table of the higher-level chapter.

The combinatorial complexity is thus bounded by the size of the largest *atomic* truth table, not by the total number of atomic concepts across all sub-chapters.

2.8.1 Concrete Example: Two-Level Hierarchy (Ṭahāra → Ṣalāh)

Let the atomic chapter **Ṭahāra** (purification for prayer) have the following binary variables (as in the Tayammum appendix):

Variable	Description	Values
h	Person has minor impurity (*ḥadath*)	$1 = \text{impure}, 0 = \text{pure}$
w	Water available	$1 = \text{available}, 0 = \text{lacking}$
u	Valid excuse (*ᶜudhr*) for tayammum	$1 = \text{excused}, 0 = \text{not excused}$
t	Prayer time entered	$1 = \text{entered}, 0 = \text{not entered}$
d	Valid dust (*ṣaᶜīd*) available	$1 = \text{available}, 0 = \text{not available}$

The complete truth table for Ṭahāra has $2^5 = 32$ rows. Running the algorithm on this table yields the minimal structural form:

$$h_{\text{ṭahāra}} = \underbrace{(h \wedge t \wedge d)}_{\text{shurūṭ}} \wedge \underbrace{(\neg w \ \vee \ u)}_{\text{candidate ᶜillah}}$$

Now consider the higher-level chapter **Ṣalāh** (the ritual prayer). The Ṣalāh chapter does not need to expand the five internal variables of Ṭahāra. Instead, it uses a single input $P = h_{\text{ṭahāra}}$ (1 if purification permits prayer, 0 otherwise). Additional local variables for Ṣalāh might be:

Variable	Description	Values
n	Intention (*niyyah*) present	$1 = \text{present}, 0 = \text{absent}$
q	Facing the *qibla*	$1 = \text{facing}, 0 = \text{not facing}$
c	Covering the *ᶜawrah*	$1 = \text{covered}, 0 = \text{not covered}$

The truth table for Ṣalāh therefore has $2^{1+3} = 16$ rows, not $2^{5+3} = 256$. The modular decomposition reduces the state space by a factor of 16 in this case.

2.8.2 Recursive Application to Deeper Hierarchies

The same pattern applies recursively. The output of the Ṣalāh chapter, $h_{\text{ṣalāh}}$, becomes a single binary variable in the **Ḥajj** truth table. Suppose Ḥajj adds two local variables (e.g., *iḥrām* properly entered, *saʿy* between Ṣafā and Marwa completed). Then the Ḥajj truth table has only $2^{1+2} = 8$ rows, as summarised below:

Level	Chapter	Local variables	Sub-chapter inputs	Total rows
1	Ṭahāra	5	0	$2^5 = 32$
2	Ṣalāh	3	1	$2^{3+1} = 16$
3	Ḥajj	2	1	$2^{2+1} = 8$

Total computational work = processing 32 rows (Ṭahāra) + 16 rows (Ṣalāh) + 8 rows (Ḥajj) = 56 rows. In contrast, a flattened single-table approach with all $5+3+2 = 10$ atomic variables would require $2^{10} = 1024$ rows and a search space of $3^{10} = 59,049$ combinations—still feasible for a computer, but significantly larger and harder for a human jurist to populate.

2.8.3 Implications for the Framework

1. **Tractability:** The effective n at any level of the hierarchy is the number of **sub-chapter outputs** plus the few **local variables** at that level. Since each sub-chapter is itself closed and small, the overall system remains computationally practical, and the $O(1)$ complexity result applies to each atomic table individually.
2. **Reusability:** Once an atomic chapter (e.g., Ṭahāra) is fully tabulated and its minimal structure extracted, the resulting binary variable can be reused across all higher chapters (Ṣalāh, Ḥajj, Ṣawm, etc.). This eliminates redundant work.
3. **Comparative madhhab analysis:** Modular decomposition allows researchers to compare madhhabs at each level independently. For example, two schools may agree on the Ṭahāra truth table but differ on the Ṣalāh truth table; the algorithm can identify where their rulings diverge.
4. **Practical construction of truth tables:** A team of jurists can populate atomic tables once, validate them against primary sources, and then compose them into larger tables without revisiting lower-level details. This turns the construction of chapter-level models into a finite and auditable formalization task, while leaving interpretive and doctrinal validation to qualified juristic scholarship.

2.8.4 Remark on Non-Modular Dependencies

Occasionally, a higher chapter may depend on a sub-chapter's *internal variables* rather than its final ruling—for example, when the ruling of Ṣalāh depends not only on whether Ṭahāra is valid, but on *why* it is valid (e.g., tayammum vs. water). In such cases, the sub-chapter's output must be refined into multiple binary flags (e.g., P_{water} and P_{tayammum}). The modular decomposition remains valid, but the interface between chapters becomes a small vector of outputs instead of a single bit. The number of such flags is still bounded by the sub-chapter's number of minimal positive rules, which is itself bounded by the small n_{sub}. Therefore, the combinatorial explosion remains contained.

3. Algorithm

We model the jurist's decision space as a Boolean truth table over the variable set V of a closed chapter. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of binary decision variables and h be the binary ruling variable. The algorithm operates in three distinct phases: Rule Extraction, Rule Shortening, and Usul Deduction.

Algorithm 1 ExtractRules(U, V)

Require: Universe of rows U, Variable set V
Ensure: Set of valid rules R

$R \leftarrow \emptyset$
for each subset $P \subset V$ of size j **do**
  for each valuation w of variables in P **do**
    $H_w \leftarrow \{h(s) \mid s \in U, s \text{ matches } w\}$
    if $|H_w| = 1$ **then**
      $R \leftarrow R \cup \{(w \Rightarrow h_w)\}$
    end if
  end for
end for
return R

Algorithm 2 ShortenRules(R)

Require: Set of valid rules R
Ensure: Set of minimal rules M

$M \leftarrow R$
for each $R_j \in R$ **do**
  for each $R_k \in M$ **do**
    if $R_k \subset R_j$ and R_k implies same ruling as R_j **then**
      Remove R_j from M
    end if
  end for
end for
return M

Algorithm 3 DeduceUsul(M+, M-)

Require: Positive minimal rules M^+, Negative minimal rules M^-
Ensure: Shurut S, 'Illah set I, Mawani' W

$S \leftarrow \bigcap_{R \in M^+} R$
$I \leftarrow \emptyset$
for each $R_i \in M^+$ **do**
  $I_i \leftarrow R_i \setminus S$
  if $I_i \neq \emptyset$ **then**
    $I \leftarrow I \cup \{I_i\}$
  else ▷ Heuristic
    $I_{\text{trigger}} \leftarrow S \setminus \{v \in S \mid v \text{ is framework variable}\}$
    $I \leftarrow I \cup \{I_{\text{trigger}}\}$
    $S \leftarrow S \setminus I_{\text{trigger}}$
  end if
end for
$W \leftarrow \emptyset$
$I_{\text{flat}} \leftarrow \bigcup_{I_k \in I} I_k$ ▷ Flatten set of candidate causes into literal set
for each $R_j \in M^-$ **do**
  if $R_j \not\subseteq \text{Inv}(S) \cup \text{Inv}(I_{\text{flat}})$ **then**
    $W \leftarrow W \cup \{R_j\}$
  end if
end for
return S, I, W

Let U denote the universe of rows of the truth table.

3.1 Rule Extraction

Generate all rules $R \Rightarrow h$ that hold universally within U.

3.2 Rule Minimization

Remove every rule that possesses a strictly smaller valid subrule. The remaining set is denoted M.

3.3 Usuli Deduction

Let M^+ be the positive minimal rules and M^- the negative minimal rules. Compute:

$$S = \bigcap_{R \in M^+} R$$

Then for each $R_i \in M^+$ compute $I_i = R_i \setminus S$. The set S represents candidate conditions, while the I_i represent candidate causes.

4. Formal Axiomatization

Axiom 11 (Shart)**.** *A variable is a Shart iff it is necessary for the ruling.*

Axiom 12 (Structural Causal Correspondence)**.** *Within a Closed Chapter, every minimal positive rule corresponds to one admissible causal pathway in the jurist's operational model.*

The role of the algorithm is to identify minimal structural generators of the ruling rather than independently establish ontological fiqh-causation.

Definition 13 (Candidate 'Illah)**.** *For a minimal positive rule R_i, the Candidate 'Illah is $I_i = R_i \setminus S$.*

Definition 14 (Inverse Literal)**.** *For a literal x, $Inv(x) = \neg x$. For a set A, $Inv(A) = \{\neg x : x \in A\}$.*

5. Correctness Results

Theorem 15 (Correctness of Shart Extraction)**.** *The algorithm correctly identifies all and only the Shurut.*

Proof. The algorithm computes $S = \bigcap_{R \in M^+} R$. A variable belongs to S iff it appears in every positive minimal rule. Therefore it is necessary for every occurrence of the ruling. Conversely, every necessary condition must appear in every positive rule. Hence S is exactly the set of Shurut. □

Theorem 16 (Correctness of Branch-Factor Extraction)**.** *For every minimal rule R_i, the algorithm correctly extracts $I_i = R_i \setminus S$.*

Proof. Since $S = \bigcap_{R \in M^+} R$, S contains precisely the universal components of all positive rules. Removing S from R_i leaves exactly the branch-specific component. Thus $I_i = R_i \setminus S$ is correctly extracted. □

Corollary 17. *Under the Structural Causal Correspondence Axiom, each I_i is an admissible Candidate 'Illah.*

Theorem 18 (Structural Classification of Negative Rules (Mani')). *Let $I_{flat} = \bigcup_i I_i$. A negative rule R_j is classified as absence-based if*

$$R_j \subseteq Inv(S) \cup Inv(I_{flat}).$$

If this inclusion fails, then R_j contains at least one literal not accounted for by the absence of an extracted condition or candidate cause. Such a literal is classified by the formal model as a candidate Mani'.

Proof. If $R_j \subseteq \mathrm{Inv}(S) \cup \mathrm{Inv}(I_{\mathrm{flat}})$, then the negative rule is explained within the formal model by the absence of an extracted condition or candidate cause. If this inclusion fails, then R_j contains at least one literal outside $\mathrm{Inv}(S) \cup \mathrm{Inv}(I_{\mathrm{flat}})$. Such a literal is not accounted for by mere absence of the extracted Shurut or candidate 'Ilal. Therefore the formal model classifies it as a candidate Mani'. □

Heuristic Principle 19 (Recovery of the Candidate 'Illah Qasirah). *When $|M^+| = 1$, the intersection operation yields $S = R_1$, and therefore $R_1 \setminus S = \emptyset$. In this special case, external semantic knowledge supplied by the jurist is required to distinguish framework variables from triggering variables. This procedure is heuristic and is not derivable from the truth table alone.*

6. Legitimacy of the Candidate 'Illah

Definition 20 (False Flag). *A false flag is a variable that co-occurs with the ruling but is logically redundant.*

Theorem 21 (Elimination of Logically Redundant Literals). *No Candidate 'Illah extracted from a minimal rule can contain a false flag.*

Proof. Assume a Candidate 'Illah contains a redundant attribute. Then there exists a smaller rule $R' \subset R_i$ that still implies the ruling. This contradicts the minimality of R_i. Therefore no extracted Candidate 'Illah contains a false flag. □

7. Co-Extensiveness

Definition 22 (Global Candidate 'Illah). *Let $I_1, \ldots, I_m$ be all Candidate 'Ilal extracted from the positive minimal rules $R_1, \ldots, R_m \in M^+$. Define $I^* = I_1 \vee I_2 \vee \cdots \vee I_m$. The formula I^* is called the Global Candidate 'Illah.*

Theorem 23 (Structural Decomposition and Co-Extensiveness). *Assume that $M^+ = \{R_1, \ldots, R_m\}$ is the complete set of positive minimal rules covering the positive region of the ruling function h. Let $S = \bigcap_{R_i \in M^+} R_i$ denote the extracted Shurut, and let $I_i = R_i \setminus S$ be the branch-specific component of each positive minimal rule. If $I^* = I_1 \vee I_2 \vee \cdots \vee I_m$, then the ruling function admits the structural decomposition $h = S \wedge I^*$. Consequently, $h \iff (S \wedge I^*)$.*

Proof. By the definition of S as the intersection of all positive minimal rules, we have $S \subseteq R_i$ for every $R_i \in M^+$. By the definition $I_i = R_i \setminus S$, each positive minimal rule decomposes as $R_i = S \wedge I_i$. Since M^+ is assumed to be the complete set of positive minimal rules covering the positive region of h, the ruling function is represented by $h = R_1 \vee R_2 \vee \cdots \vee R_m$. Substituting $R_i = S \wedge I_i$ gives $h = (S \wedge I_1) \vee (S \wedge I_2) \vee \cdots \vee (S \wedge I_m)$. By distributivity of conjunction over disjunction, this becomes $h = S \wedge (I_1 \vee I_2 \vee \cdots \vee I_m)$. By Definition 22, we have $I^* = I_1 \vee I_2 \vee \cdots \vee I_m$. Therefore $h = S \wedge I^*$. Hence $h \iff (S \wedge I^*)$. □

8. Structural Soundness

Theorem 24 (Structural Soundness)**.** *Given: (1) a Closed Chapter, (2) a complete variable set V, and (3) a complete truth table, the algorithm computes exactly the subset-minimal implicating partial valuations for the ruling function h.*

Proof. Algorithm 1 enumerates all valid implicating rules. Algorithm 2 removes every non-minimal rule. The remaining set M is exactly the set of subset-minimal implicating partial valuations for h. All subsequent classifications are computed from this set. □

9. Semantic Boundary

The algorithm establishes structural legitimacy but not semantic suitability (*munasabah*). If irrelevant variables are supplied, the algorithm will correctly identify their structural role within the truth table, but cannot determine whether they reflect the objectives of the Lawgiver. That responsibility remains with the faqih.

10. Computational Complexity and Tractability

A rigorous analysis of the algorithm's complexity requires distinguishing between the general computational complexity of exhaustive rule extraction and the resource constraints imposed by the Closed Chapter Assumption. We demonstrate that while the algorithm performs a task that is classically exponential in the number of variables, a fixed closed chapter yields a bounded finite computation whose upper bound depends only on the size of that chapter's concept set.

Theorem 25 (Bounded Resource Consumption for a Fixed Closed Chapter)**.** *For any fixed closed chapter with a fixed concept set of size k, the algorithmic pipeline has a constant upper bound depending only on k.*

Remark 26. *This is not a polynomial-time claim in an unbounded variable parameter n; rather, it is a bounded-domain claim relative to a fixed closed chapter.*

Proof. Unlike the Quine-McCluskey algorithm, which iteratively combines adjacent minterms bottom-up, Algorithm 1 operates top-down via exhaustive subset-valuation projection. The mechanism proceeds as follows:

1. Generate all possible subsets $s \subset V$ of the variable set.
2. For a given subset s of size j, generate all 2^j possible valuations w of the variables in s.
3. Project the valuation w onto the truth table U. If all rows matching the valuation w yield a uniform ruling (i.e., the projection reduces to a single value), then deduce the rule $r : (w \Rightarrow h_w)$.
4. Accumulate all such valid rules into the rule set R.

Algorithm 2 subsequently minimizes this set by removing any rule strictly subsumed by a smaller valid rule. The resulting set M is mathematically equivalent to the set of prime implicants of the ruling function [2, 3], though derived via a different algorithmic pathway.

To determine the complexity, we evaluate the number of subset-valuation combinations Algorithm 1 must project. For a variable set of size n, the total number of valuations evaluated across all subsets is:

$$\sum_{j=0}^{n} \binom{n}{j} 2^j = (1+2)^n = 3^n.$$

This operation is classically exponential with respect to the number of variables, n. If n were unbounded, the algorithm would require exponential resources.

However, by the Closed Chapter Assumption, the legally operative concept set V is finite and fixed for a given Madhhab. Therefore, the number of variables is strictly bounded such that $n = k$ for some constant $k > 0$.

Because n is a fixed constant k for the chapter under investigation, we can establish strict constant upper bounds for all parameters in the algorithm:

1. The number of subset-valuation combinations evaluated by Algorithm 1 is exactly 3^k. For each combination, projecting and checking for a uniform ruling requires at most $O(n)$ operations. Thus, the runtime of Algorithm 1 is $O(3^k \cdot k) = O(1)$.

2. The number of rules generated by Algorithm 1 is $m = |R|$. Since Algorithm 1 generates at most one rule per evaluated valuation, $m \leq 3^k$.

3. The number of minimal rules output by Algorithm 2 is $p = |M^+| + |M^-|$. Since Algorithm 2 only removes rules from R to form M, it follows that $p \leq m \leq 3^k$.

Substituting these bounded parameters into the complexity of the remaining algorithms:

- Algorithm 2 runs in $O(m^2 \cdot k) \leq O((3^k)^2 \cdot k) = O(9^k \cdot k) = O(1)$.
- Algorithm 3 runs in $O(p \cdot k) \leq O(3^k \cdot k) = O(1)$.

Since all parameters governing the loops and operations across all three algorithms depend exclusively on the constant k, relative to variation outside the fixed chapter, the total time complexity of the pipeline is bounded by a constant depending only on k. Thus, for a fixed closed chapter, the top-down projection of all variable subsets is bounded by a constant depending only on that chapter's fixed concept set. □

Remark 27 (Relation to Quine-McCluskey Minimization)**.** *While Algorithm 1 and Algorithm 2 jointly compute the set of prime implicants—mathematically equivalent to the output of the Quine-McCluskey algorithm—the top-down subset-valuation projection method is better suited for the formalization of Fiqh. Quine-McCluskey is not by itself tailored to the present juristic task for three reasons:*

1. ***Multi-Valued Rulings (Aḥkām):*** *Quine-McCluskey is strictly designed for binary Boolean functions ($h \in \{0, 1\}$). In Fiqh, the ruling variable h often takes on multiple discrete values (e.g., Wājib, Mandūb, Mubāḥ, Makrūh, Ḥarām). Algorithm 1 naturally extracts rules for any uniform ruling category by simply checking if the projection H_w contains a single value, regardless of the cardinality of the ruling set.*

2. ***Symmetric Extraction of Mawāni' (Preventatives):*** *Quine-McCluskey is optimized to minimize the positive cover (the "ON-set") using "don't-care" states to simplify the circuit. Fiqh admits no "don't-care" states; the negative space must be explicitly analyzed to extract Mawāni' (independent preventatives). Algorithm 1 symmetrically extracts both positive (M^+) and negative (M^-) minimal rules, which is a strict prerequisite for the Usulī deduction of Mawāni' in Algorithm 3.*

3. ***Preservation of Causal Pathways:*** *Quine-McCluskey combines minterms bottom-up, which optimizes circuit logic but obscures the distinct causal pathways (ṭuruq al-ᶜillah). The top-down projection of Algorithm 1 preserves the disjunctive normal form structure necessary to separate the Shurūṭ (S) from the branch-specific 'Ilal (I_i), directly mapping the output to the epistemic structure of al-Ṣabr wa al-Taqsim.*

Remark 28. *The significance of the preceding bounded-resource theorem is that it cleanly separates the theoretical limits of Boolean logic from the operational reality of Fiqh. In digital circuit design, n can grow infinitely (e.g., $n > 50$), rendering exhaustive subset projection computationally intractable. In Fiqh, the Closed Chapter Assumption ensures n remains small and strictly bounded by the text of the Madhhab (typically $n \leq 10$ for a single legal chapter). The epistemological closure of the chapter mathematically guarantees the computational tractability of the algorithm.*

11. Conclusion

Given a complete truth table for a Closed Chapter, the algorithm computes the minimal structural generators of the ruling, eliminates all logically redundant attributes, and produces formally admissible candidate *'ilal*. This is illustrated in the appendix through a reduced version of the OCT for the Tahara chapter of the Shafi'i school. The algorithm therefore constitutes a rigorous computational analogue of al-Ṣabr wa al-Taqsim, while leaving semantic validation to juristic expertise.

A natural question concerns the range of applicability of the present framework. The framework does not claim that every chapter of fiqh is necessarily a Closed Chapter, nor that the complete conceptual vocabulary of every legal domain has already been discovered. Rather, it establishes that whenever a chapter can be represented as a conceptually closed operational model—namely, when all legally operative concepts have been identified and all future cases can be expressed using that fixed conceptual vocabulary—the method of *al-Ṣabr wa al-Taqsim* admits an exact computational formalization through complete induction over a finite state space. The validity of the mathematical results therefore depends not upon the universal truth of the Closed Chapter Assumption, but upon its satisfaction within the specific chapter under investigation. Consequently, the framework should be understood as a conditional theory: if conceptual closure holds, then the extraction of structurally admissible candidate causes reduces to the computation of subset-minimal implicating partial valuations of the corresponding ruling function.

A. Computational Usuli Analysis: Tayammum (Shafi'i School)

بسم الله الرحمن الرحيم

تحليل أصولي حاسوبي لمسألة: التيمم (المذهب الشافعي)

المستخرج آلياً من جدول الحقائق المنطقي

القسم الأول: جدول تصور المسائل (جدول الحقائق)

تم ضبط عرض الأعمدة وحجم الخط لضمان احتواء الجدول بالكامل ضمن هوامش الصفحة دون تجاوزها.

المحدث	الماء	العذر	الوقت	الأداة	الحكم
المكلف طاهر	واجد الماء	غير معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	واجد الماء	غير معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	واجد الماء	غير معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	واجد الماء	غير معذور	دخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	واجد الماء	معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	واجد الماء	معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	واجد الماء	معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	واجد الماء	معذور	دخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	فاقد الماء	غير معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	فاقد الماء	غير معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	فاقد الماء	غير معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	فاقد الماء	غير معذور	دخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	فاقد الماء	معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	فاقد الماء	معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف طاهر	فاقد الماء	معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف طاهر	فاقد الماء	معذور	دخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	واجد الماء	غير معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	واجد الماء	غير معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	واجد الماء	غير معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	واجد الماء	غير معذور	دخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	واجد الماء	معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	واجد الماء	معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	واجد الماء	معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	واجد الماء	معذور	دخل الوقت	الأداة صالحة	**جائز**
المكلف محدث	فاقد الماء	غير معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	فاقد الماء	غير معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	فاقد الماء	غير معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	فاقد الماء	غير معذور	دخل الوقت	الأداة صالحة	**جائز**
المكلف محدث	فاقد الماء	معذور	لم يدخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	فاقد الماء	معذور	لم يدخل الوقت	الأداة صالحة	غير جائز
المكلف محدث	فاقد الماء	معذور	دخل الوقت	الأداة غير صالحة	غير جائز
المكلف محدث	فاقد الماء	معذور	دخل الوقت	الأداة صالحة	**جائز**

القسم الثاني: القواعد الفقهية المجملة (القواعد المستخلصة)

أولاً: قواعد الإثبات (متى يكون الحكم جائزاً/مثبتاً):

1. (المكلف محدث و فاقد الماء و دخل الوقت و الأداة صالحة) يقتضي حكماً: جائز
2. (المكلف محدث و معذور و دخل الوقت و الأداة صالحة) يقتضي حكماً: جائز

ثانياً: قواعد النفي (متى يكون الحكم غير جائز/منفياً):

1. (المكلف طاهر) يقتضي حكماً: غير جائز
2. (لم يدخل الوقت) يقتضي حكماً: غير جائز
3. (الأداة غير صالحة) يقتضي حكماً: غير جائز
4. (واجد الماء و غير معذور) يقتضي حكماً: غير جائز

القسم الثالث: التحليل الأصولي (استنباط الشروط والعلة والموانع)

1. **الشروط (مقومات الحكم):**
   - المكلف محدث
   - الأداة صالحة
   - دخل الوقت

[وجه الاستنباط]: لأنها تلازم الحكم في جميع صوره المثبتة.

2. **العلة (المناط المؤثر في الحكم):**
   تعددت طرق العلة، وهي:
   - فاقد الماء
   - معذور

[وجه الاستنباط]: لأنها الوصف المؤثر الذي تتعدد طرقه وتوجب الحكم.

3. **انتفاء الشروط والعلل (أسباب عدم الحكم):**
   - انتفاء شرط: المكلف طاهر
   - انتفاء شرط: لم يدخل الوقت
   - انتفاء شرط: الأداة غير صالحة
   - انتفاء العلة: غير معذور و واجد الماء

[وجه الاستنباط]: لأن انتفاء الشرط أو العلة غياب لمقومات الحكم لا مانع مستقل.

4. **الموانع المستقلة (ما يمنع الحكم مع تحقق الشروط والعلل):**
   - لا يوجد مانع مستقل في هذه المسألة.

[وجه الاستنباط]: لا يوجد مانع مستقل عن انتفاء الشروط أو العلة.

انتهى التحليل برب التوفيق

Statements and Declarations

Competing Interests

The authors have no competing interests to declare.

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Discussion Note

Cantor, Axiomatic Set Theory, and the Articulation of Multiplicity

A Note on Metric Abstraction and Zero-Dimensional Continua

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ABSTRACT

A standard construction in fractal geometry yields Cantor-type sets of Hausdorff dimension zero whose cardinality remains that of the continuum. This discussion note uses that familiar limit case as a clarifying exhibit rather than as a source of mathematical novelty. The central claim is that the case helps distinguish metric presentation from formal multiplicity and, on that basis, sharpens a philosophical demarcation between Cantor's conception of set and later axiomatic set theory. Cantor's account of set and magnitude still keeps visible the gathering of many into one whole, whereas axiomatic set theory supplies a formal articulation of multiplicity once identity, distinguishability, and collection are already available. The zero-dimensional continuum case renders one aspect of that contrast especially clear. A brief final section indicates one consequence for the structuralist identity problem.

Keywords: continuum, Hausdorff dimension, Cantor set, abstraction, set theory, unity, structuralism

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1. Introduction

Classical mathematics often coordinates geometric space and arithmetic multiplicity so smoothly that their conceptual distinction recedes from view. In ordinary geometric imagination, continuum multiplicity appears together with spread, thickness, and extension. This note studies a mathematically controlled limit case in which a standard metric notion of scaling reaches an extremum while continuum cardinality remains unchanged. The significance of the case extends beyond fractal geometry. It helps clarify the difference between metric presentation and formal multiplicity.

The philosophical pressure behind this inquiry is familiar. Neo-Aristotelian concerns about composing a continuum from extensionless indivisibles continue to function as a recurring touchstone in discussions of continuity and divisibility [White, 1992]. Weyl likewise presents the continuum as bound up with a form of intuition or "given" extension in the background conception of analysis [Weyl, 1949]. These positions register a broader family of philosophical habits according to which continuum multiplicity travels together with some robust notion of geometrical or metric spread. The present note identifies a standard mathematical case that distinguishes those levels with particular clarity.

The mathematical exhibit is familiar. Fractal geometry provides Cantor-type constructions whose Hausdorff dimension is zero although their cardinality remains that of the continuum [Falconer, 1985, Example 4.6]. The purpose of the example here is not to establish new mathematics. Its role is diagnostic. It offers a controlled setting in which one central mode of geometric presentation recedes while full formal multiplicity remains available. That observation in turn helps articulate a philosophical distinction that is often present in the literature without always being stated sharply: the distinction between Cantor's many-into-one conception of set and later axiomatic set theory as a formal articulation of multiplicity.

A related methodological orientation already appears in Cantor's remarks on the continuity of space. Cantor treats the usual continuity hypothesis as an optional representational identification: a complete one–one correspondence between the three-dimensional arithmetical continuum (x, y, z) and the concept of space used to describe appearances. As he writes, such an identification proceeds as a "free act" of constructive thought rather than through "inner compulsion" [Cantor, 1932, pp. 156–157]. Together with Cantor's explicit abstraction from nature and order [Cantor, 1915, ğ 2], this provides the historical point of departure for the extension pursued here.

Cantor's own set-theoretic framework is often characterized by a two-step abstraction: abstraction from the nature of the elements, and abstraction from the order in which they are given. In that spirit, this note introduces what it will call the *Metric Abstraction*: the methodological step of considering the cardinality of a set in abstraction from its metric realization. The phrase names no new mathematical operation. It names a way of describing, in the present context, a familiar separation between metric and set-theoretic features.

The argument proceeds in five steps. Section 1 reconstructs Cantor's abstraction program and identifies the role of the gathered whole within it. Section 2 presents the generalized Cantor construction and the zero-dimensional continuum result. Section 3 interprets that result as a case of metric abstraction and separation between spread and multiplicity. Section 4 draws a demarcation between Cantor and axiomatic set theory. Section 5 formulates the relation between gathered whole and formal articulation more directly. A brief final section indicates one consequence for the structuralist identity problem.

2. Cantor's Abstraction Program

Cantor's original conception of set theory stands close to the problem of unity. A set, in his classical formulation, is a collection of definite and well-distinguished objects gathered into one whole [Cantor, 1895, 1915, 1932]. The phrase "into one whole" is philosophically decisive. It indicates that sethood enters together with a unifying act through which multiplicity is held together as one domain.

Within this conceptual field, Cantor's account of magnitude proceeds by abstraction. First, one abstracts from the particular nature of the elements. Second, one abstracts from the order in which they are given [Cantor, 1915, ğ 2]; see also [Cantor, 1932; Hallett, 1984]. The result of this double abstraction is cardinality or *Mächtigkeit*. In this sense, cardinality names a residue of abstraction. Yet that residue still belongs to a framework in which the many are gathered into one whole.

This point matters for the present discussion because it keeps visible a level that later formal accounts often leave implicit. Cantor's program aims at a purified account of multiplicity while preserving the conceptual role of the gathered whole. Set and magnitude remain related to a unifying domain. His framework therefore occupies a threshold position: it advances abstraction powerfully and still keeps visible the passage from many to one.

The present note extends this line of thought in a limited way. If abstraction from nature and order yields cardinality, further determinations also become eligible for abstraction. The additional step considered here is metric abstraction. The aim is not to revise Cantor historically, but to use his abstraction program as a philosophical point of departure.

3. The Zero-Dimensional Continuum Case

There is a well-known construction in fractal geometry that yields a Cantor-type set with Hausdorff dimension zero while preserving cardinality $\mathfrak{c}$ [Falconer, 1985, Example 4.6]. We use this standard variable dissection construction as the mathematical exhibit for the philosophical argument developed below.

Consider a non-stationary Cantor construction C_{k_n} in which the contraction factor $k_n \geq 3$ varies with the stage n. Using a standard Hausdorff-measure covering estimate, the scaling ratio

$$R_n = \prod_{j=1}^{n} \frac{1}{k_j}$$

can be driven to a limit at which the Hausdorff scaling profile vanishes. As shown in Appendix A, choosing a super-exponential sequence such as $k_n = 3^{n!}$ yields

$$\dim_H(C_{k_n}) = 0.$$

The same construction also preserves a binary branching pattern at every stage. Each point of C_{k_n} determines a symbolic address

$$\omega = (\omega_1, \omega_2, \dots) \in \{0,1\}^{\mathbb{N}}.$$

To resolve dual representations at stage-endpoints, we adopt the standard convention of choosing the address that is not eventually constant equal to 1 [Edgar, 2008, p. 31]. This yields a canonical coding.

Let $\Omega \subseteq \{0,1\}^{\mathbb{N}}$ denote the set of canonical addresses. The addressing map induces a bijection

$$\phi : C_{k_n} \longrightarrow \Omega.$$

Since Ω differs from $\{0,1\}^{\mathbb{N}}$ by only a countable subset, it has cardinality $\mathfrak{c}$. The construction therefore yields a precise configuration in which a standard metric scaling invariant is zero while set-theoretic multiplicity remains that of the continuum.

The mathematical significance of the example lies in the explicit coordination of two levels: vanishing metric scaling and preserved continuum multiplicity. For present purposes, that coordination makes the case a useful limit exhibit.

4. Metric Abstraction and the Separation of Spread from Multiplicity

The philosophical interest of Hausdorff dimension arises from its precise role as a metric scaling invariant. Set-theoretic cardinality tracks pure multiplicity. Topological properties track neighborhood structure and continuity relations. Hausdorff dimension captures the exponent governing how the number of ε-balls needed to cover a set grows as $\varepsilon \to 0$, equivalently, which power-law scalings yield nontrivial Hausdorff measure [Rogers, 1970; Falconer, 1985]. In this note, the phrase "metric extension" refers to this scaling profile.

Accordingly, when $\dim_H(C_{k_n}) = 0$, the case displays an extremal form of metric thinness together with full continuum multiplicity. The set continues to carry rich branching structure, and its cardinality remains that of the continuum. The example therefore isolates a precise form of multiplicity-with-vanishing-metric-scaling: continuum many positions remain available while the Hausdorff scaling exponent reaches zero.

The philosophical point is limited but useful. Metric spread and multiplicity belong to distinct representational levels. The example renders that distinction especially visible by displaying the two features in maximal separation. The point itself is familiar; the present value of the case lies in the sharpness with which it displays the contrast.

The address structure makes this especially clear. When metric scaling is attenuated to a Hausdorff-dimension-zero limit, the structure relevant to tracking points is preserved through the infinite binary strings that encode branching choices. In this setting, the address space Ω functions as the canonical combinatorial invariant by which multiplicity is represented.

The same case also illustrates a second distinction. At the level of addressable representation, one can speak of continuum many distinct positions. Yet these positions belong to one globally given structure. They are differentiations within one support. The limit case therefore serves as a controlled exhibit for discussing the relation between gathered whole and articulated multiplicity.

5. Cantor and Axiomatic Set Theory

The zero-dimensional continuum case now permits a sharper statement of a familiar philosophical demarcation between Cantor's conceptual program and later axiomatic set theory.

Cantor's account remains tied to a gathered whole. His language of collection, abstraction, and magnitude still concerns the passage from many to one whole. Even where abstraction strips away nature and order, the many are already being gathered. Unity remains conceptually visible within the formation of magnitude.

Axiomatic set theory operates at a different level. It gives a formal calculus of membership, hierarchy, and collection once admissible objects, identity, and distinguishability are already available. It articulates multiplicity with great precision. Its formal strength is unmistakable. Its starting point, however, belongs to a level at which the gathered whole is already in place.

This is the central demarcation of the note. Cantor's program is a theory of abstraction toward magnitude within a gathered whole. Axiomatic set theory is a calculus of articulated multiplicity once that gathered-whole level has already been presupposed. The later formalism preserves the articulation of multiplicity while the unifying moment moves into the background.

The zero-dimensional continuum case does not establish this demarcation by itself. It renders one aspect of it especially clear. It shows that full formal multiplicity can remain available where a familiar mode of geometric presentation has receded to an extremal limit. That makes the distinction between gathered whole and formal articulation easier to isolate.

6. Representation and Gathered Whole

The preceding distinction leads to a more exact question about philosophical architecture. Set theory represents unity and multiplicity with great formal power. The question concerns the level at which that representation begins.

Consider the standard set-theoretic account of 1. Formally, one may define

$$0 = \varnothing, \qquad 1 = \{\varnothing\}.$$

This definition is fully serviceable within the formalism. Its philosophical significance is derivative. It expresses oneness through prior objecthood and prior collection. The empty set appears as an admissible object; the singleton appears as a gathered whole containing that object. The definition therefore locates one within a formal articulation of already available determinations.

The same point extends more broadly. Set theory proceeds with variables, identity, distinguishability, and membership already available. It thereby provides a formal grammar of articulated multiplicity. The unity through which objects are first gathered and distinctions first become available belongs to a prior level of description.

The present note treats this as a matter of philosophical architecture rather than as a complaint about formal adequacy. Set theory articulates the many with great exactness. Cantor's language keeps visible the gathered whole within which such articulation proceeds. The difference between those two levels is the main point at issue here.

The zero-dimensional continuum case gives this distinction a vivid if modest illustration. It shows that full formal multiplicity can remain where metric spread vanishes. The example thereby clarifies the relation between support and articulation in a controlled setting without asking the mathematics to settle the broader philosophical issue on its own.

7. A Brief Consequence for the Structuralist Identity Problem

Once the distinction between support and articulation is in place, the structuralist identity problem appears in a revised light [Keränen, 2001; Shapiro, 2008]. The present note makes only a limited point here.

In the zero-dimensional continuum case, each point of C_{k_n} is canonically associated with an address in Ω. The address space thereby provides an interface through which distinct items in the target domain are represented and re-identified. Mathematical individuation proceeds through canonical relational coding together with a globally given support.

This suggests a more exact location for the problem. One may distinguish:

1. the underlying structure,
2. the interface through which that structure becomes mathematically accessible and trackable,
3. and the background resources through which the distinctness of interface elements is sustained.

In the present case, C_{k_n} supplies the underlying support, while the canonical address space Ω supplies the interface through which points of that support are accessed and re-identified. The note's claim is simply that this distinction helps locate the identity problem more precisely. The broader structuralist debate remains exactly where it is.

8. Conclusion

This note has used a standard fractal construction to isolate a limit case in which a metric scaling invariant, Hausdorff dimension, vanishes while set-theoretic cardinality remains $\mathfrak{c}$. The resulting example brings into focus one precise sense in which metric presentation and formal multiplicity come apart: continuum many positions remain available while the Hausdorff scaling profile reaches zero.

The significance of that case is philosophical rather than mathematical. It helps articulate a demarcation between Cantor's abstraction program and later axiomatic set theory. Cantor still stands near the passage from many to one whole. Axiomatic set theory formalizes articulated multiplicity once gathered whole, identity, and collection are already available.

The broader lesson therefore concerns the distinction between gathered whole and formal articulation. Set theory gives a formal grammar of the many, while Cantor's formulation keeps visible the unity within which multiplicity is gathered. The zero-dimensional continuum case serves as a useful exhibit for that distinction.

A. Hausdorff Dimension Estimate

At stage n, C_{k_n} is covered by 2^n intervals of length R_n. For $s > 0$, the Hausdorff s-measure obeys

$$\mathcal{H}^s(C_{k_n}) \leq \liminf_{n\to\infty} 2^n R_n^s = \liminf_{n\to\infty} \exp\left(n \ln 2 - s \sum_{j=1}^{n} \ln k_j \right).$$

Given the super-exponential sequence $k_n = 3^{n!}$, the ratio

$$\frac{n \ln 2}{\sum_{j=1}^{n} \ln k_j}$$

vanishes in the limit. Hence for any fixed $s > 0$, the exponent tends to $-\infty$, so the covering bound tends to 0. Therefore

$$\mathcal{H}^s(C_{k_n}) = 0 \quad \text{for all } s > 0,$$

which implies

$$\dim_H(C_{k_n}) = 0$$

[Carathéodory, 1914; Rogers, 1970; Falconer, 1985].

Statements and Declarations

Competing Interests

The author declares no competing interests.

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